Two-Moment Inequalities for Rényi Entropy and Mutual Information

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How can we show mutual information is small?

1. Use $I(X;Y) = H(X) - H(X \,|\, Y)$

2. Jensen's inequality and Rényi divergence:

   $$I(X;Y) = D_1(P_{X,Y} \| P_X P_Y) \leq D_\alpha(P_{X,Y} \| P_X P_Y), \qquad \alpha > 1$$

3. A 'half Jensen' inequality and a 'two-moment' inequality
   Today's talk

Motivation: Conditional CLT for random projections

Consider $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \sqrt{t}\boldsymbol{N}$ where $\boldsymbol{A}$ is IID Gaussian random matrix and $\boldsymbol{N}$ is Gaussian perturbation. Let $G_{\boldsymbol{Y}}$ be Gaussian distribution with same mean and covariance as $\boldsymbol{Y}$.

$$P_{\boldsymbol{Y}} \approx G_{\boldsymbol{Y}} \qquad \text{CLT}$$

$$P_{\boldsymbol{Y}|\boldsymbol{A}}(\cdot \,|\, \boldsymbol{A}) \approx G_{\boldsymbol{Y}} \qquad \text{Conditional CLT}$$

To prove entropic bounds (see Friday's talk), we use

$$\{I(\boldsymbol{A};\boldsymbol{Y}) \approx 0\} \quad \text{and} \quad \text{CLT} \quad \Longleftrightarrow \quad \text{Conditional CLT}$$

Challenges:

- tight bounds on $h(\boldsymbol{Y})$ and $h(\boldsymbol{Y} \,|\, \boldsymbol{A})$ are difficult
- $p(\boldsymbol{Y} \,|\, \boldsymbol{A})/p(\boldsymbol{Y})$ can increase without bound as $t \to 0$

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Jensen and Rényi divergence revisited

For random variables $(X, Y) \sim p(x, y)$,

$$I(X; Y) = \mathbb{E}[\log Z], \qquad Z = \frac{p(X, Y)}{p(X)p(Y)}$$

Jensen's inequality

$$I(X; Y) = \mathbb{E}[\log Z] = \frac{1}{t}\mathbb{E}\left[\log Z^t\right] \leq \frac{1}{t}\log \mathbb{E}\left[Z^t\right]$$

This is the Rényi divergence of order $\alpha = 1 + t$.

This approach is problematic if Z has heavy tails...

A 'half Jensen' inequality [R. 2017]

Start with 'half Jensen' inequality

$$I(X;Y) = \mathbb{E}[\log Z] \leq \mathbb{E}[\log(\mathbb{E}[Z \,|\, Y])]$$

Conditional expectation of Z can be expressed in terms of the variance of conditional density:

$$\mathbb{E}[Z \,|\, Y = y] = \int \left[\frac{p(y \,|\, x)}{p(y)} \right]^2 p(x) \, \mathrm{d}x = 1 + \frac{\mathsf{Var}(p(y \,|\, X))}{p(y)}$$

For every $0 < t < 1$, the inequality $\log(1+u) \leq \frac{1}{t} u^t$ yields,

$$I(X;Y) \leq \frac{1}{t} \int [p(y)]^{1-2t} [\mathsf{Var}(p(y \,|\, X))]^t \, \mathrm{d}y$$

Special cases of the 'half Jensen' inequality

$$I(X;Y) \leq \frac{1}{t} \int [p(y)]^{1-2t} [\mathsf{Var}(p(y \,|\, X))]^t \, \mathrm{d}y$$

The case $t = 1$ gives bound in terms of chi-square divergence

The case $t = 1/2$ gives bound in terms of standard deviation of the conditional density

$$I(X;Y) \leq 2 \int \sqrt{\mathsf{Var}(p(y \,|\, X)} \, \mathrm{d}y$$

The case $0 < t < 1/2$ combined with Hölder's inequality gives bound in terms of variance of the conditional density and Rényi entropy of order $r = (1-2t)/(1-t)$

$$I(X;Y) \leq \frac{1}{t} \left[\exp(h_r(Y)) \int \mathsf{Var}(p(y \,|\, X) \, \mathrm{d}y \right]^t$$

These bounds depend on integrals of fractional powers.

A 'two-moment' inequality [R. 2017]

Proposition: For any numbers $0 < r < 1$ and $p < 1/r - 1 < q$ and non-negative function f defined on $[0, \infty)$, we have

$$\underbrace{\left(\int |f(x)|^r \, \mathrm{d}x \right)^{\frac{1}{r}}}_{\|f\|_r} \leq C \left(\underbrace{\int \|x\|^p f(x) \, \mathrm{d}x}_{p\text{-th moment}} \right)^{\lambda} \left(\underbrace{\int \|x\|^q f(x) \, \mathrm{d}x}_{q\text{-th moment}} \right)^{1-\lambda}$$

where $\lambda = (q + 1 - 1/r)/(q - p)$.

Best possible constant is given by

$$C = \left[\frac{1}{(q-p)} \widetilde{\mathrm{B}} \left(\frac{r\lambda}{1-r}, \frac{r(1-\lambda)}{1-r} \right) \right]^{\frac{1-r}{r}}$$

with $\widetilde{B}(a, b) = \mathrm{B}(a, b)(a + b)^{a+b} a^{-a} b^{-b}$.

A ‘two-moment’ inequality [R. 2017]

Proposition: For any numbers $0 < r < 1$ and $p < 1/r - 1 < q$ and non-negative function f defined on $S \subseteq \mathbb{R}^n$, we have

$$\underbrace{\left(\int |f(x)|^r \, \mathrm{d}x \right)^{\frac{1}{r}}}_{\|f\|_r} \leq C \left(\underbrace{\int \|x\|^{np} f(x) \, \mathrm{d}x}_{np\text{-th moment}} \right)^{\lambda} \left(\underbrace{\int \|x\|^{nq} f(x) \, \mathrm{d}x}_{nq\text{-th moment}} \right)^{1-\lambda}$$

where $\lambda = (q + 1 - 1/r)/(q - p)$.

Best possible constant is given by

$$C = \left[\frac{\mathrm{Vol}(B^n \cap \mathrm{cone}(S))}{(q-p)} \widetilde{\mathrm{B}} \left(\frac{r\lambda}{1-r}, \frac{r(1-\lambda)}{1-r} \right) \right]^{\frac{1-r}{r}}$$

with $\widetilde{B}(a,b) = \mathrm{B}(a,b)(a+b)^{a+b} a^{-a} b^{-b}$.

Remarks on the ‘two-moment’ inequality

The proof is a straightforward consequence of Hölder’s inequality and integral representation of the Beta function.

For $r = 1/2$, Euler’s reflection formula for the Beta function leads to simplified expression

$$C = \frac{\pi \lambda^{-\lambda} (1-\lambda)^{-(1-\lambda)}}{(q-p)\sin(\pi\lambda)}$$

It is possible that variations of these inequalities exist in the literature on weighted L^p-norm inequalities. So far, I have been unable to find an explicit reference.

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Rényi entropy

Let X have density $p(x)$ on $S \subseteq \mathbb{R}^n$. The Rényi entropy of order $r \in (0,1) \cup (1,\infty)$ is defined according to

$$h_r(X) = \frac{1}{1-r} \log\left(\int_S |p(x)|^r \, \mathrm{d}x \right).$$

Properties:

- Decreasing in r
- Limit as $r \to 0$ depends on volume of support
- Limit as $r \to 1$ is Shannon entropy

Upper bound for Rényi entropy [R. 2017]

Proposition: For any numbers $0 < r < 1$ and $p < 1/r - 1 < q$ and density function $p(x)$ on $S \subseteq \mathbb{R}^n$, we have

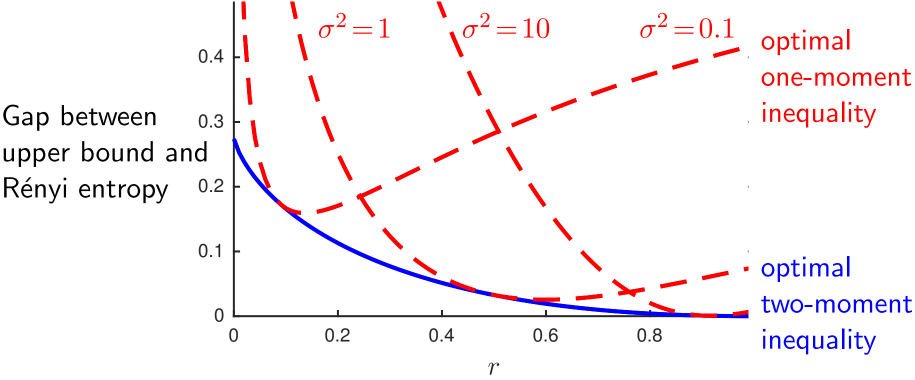
$$h_r(X) \le \log \widetilde{C} + \frac{r\lambda}{1-r} \log \mathbb{E}[\|X\|^{np}] + \frac{r(1-\lambda)}{1-r} \log \mathbb{E}[\|X\|^{nq}]$$

with $\lambda = (q + 1 - 1/r)/(q - p)$.

Relationship to existing results:

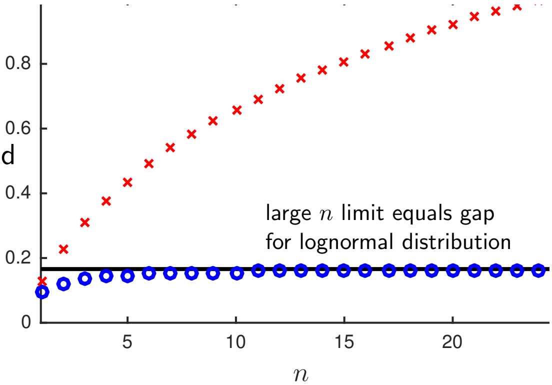
- **One-moment bound:** Evaluating with $p = 0$ recovers entropy–moment inequalities of Costa et al. 2002 and Lutwak et al. 2002. Bound is attained by a maximum entropy distribution.
- **Two-moment bound:** Evaluating with two carefully chosen moments (p, q) can lead to significant improvements.

Example: Lognormal distribution



Example: Multivariate Gaussian distribution

Gap between upper bound and Rényi entropy ($r = 0.2$)



optimal one-moment inequality

optimal two-moment inequality

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Variance of conditional density

Let (X, Y) be a random pair such that the conditional distribution of Y given X has density $p(y\,|\,x)$ on $\mathbb{R}^n$.

Variance of conditional density defined by

$$\mathsf{Var}(f(y\,|\,X)) = \mathbb{E}\Big[\big|p(y\,|\,X) - p(y)\big|^2\Big], \qquad X \sim P_X$$

Define s-th moment of the variance:

$$V_s(Y\,|\,X) = \int \|y\|^s \,\mathsf{Var}(p(y\,|\,X))\,\mathrm{d}y.$$

Note that $V_s(Y|X)$ is nonnegative and equal to zero if and only if X and Y are independent.

A mutual information bound [R. 2017]

Combining the 'half Jensen' and 'two-moment' inequalities yields:

$$I(X;Y) \leq C_\lambda \sqrt{\frac{\omega(S) V_{np}^{\lambda}(Y|X) V_{nq}^{1-\lambda}(Y|X)}{(q-p)}}, \qquad q < 1 < p$$

where $\lambda = (q-1)/(q-p)$ and $\omega(S) = \mathrm{Vol}(B^n \cap \mathrm{cone}(S))$

Useful properties of $V_s(Y \,|\, X)$

The s-th moment of the variance can be expressed as

$$V_s(Y \,|\, X) = \mathbb{E}[K_s(X, X) - K_s(X_1, X_2)],$$

where X_1 and X_2 are independent and

$$K_s(x_1, x_2) = \int \|y\|^s f(y|x_1) f(y|x_2)\, \mathrm{d}y$$

is a p.d. kernel that does not depend on the distribution of X.

If $U \to X \to Y$ forms a Markov chain then

$$V_s(Y \,|\, U) = \mathbb{E}\big[K_s(X_1', X_2') - K_s(X_1, X_2)\big]$$

where X_1' and X_2' are conditionally independent given U

Illustrative example

Consider Markov chain $U \to X \to Y$ given by

$$X \mid U \sim \mathcal{N}(0, U), \qquad Y \mid X \sim \mathcal{N}(X, 1).$$

Then

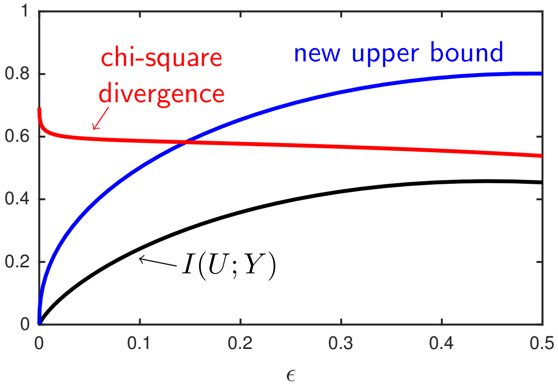
$$K_s(x_1, x_2) = 2^{-\frac{1+s}{2}} \mathbb{E}\left[\left|\mathcal{N}(0,1) + \frac{x_1 + x_2}{\sqrt{2}}\right|^s\right] \phi\left(\frac{x_1 - x_2}{\sqrt{2}}\right),$$

and

$$V_s(Y|U) = \frac{\Gamma(\frac{1+s}{2})}{2\pi} \mathbb{E}\left[(1+U)^{\frac{s-1}{2}} - \frac{(1+U_1)^{\frac{s}{2}}(1+U_2)^{\frac{s}{2}}}{(1+\frac{1}{2}(U_1+U_2))^{\frac{s+1}{2}}}\right],$$

where (U_1, U_2) are independent copies of U.

Illustrative example



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Conclusion

- Bounds on mutual information using a 'half Jensen' inequality and a 'two-moment' inequality.
- Two carefully chosen moments can lead to significant improvements for Rényi entropy.
- New measure of dependence with interesting properties.
- One application is precise convergence rates for entropic conditional central limit theorem [R. 2017]

References I

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