9.1 Motivation: Quantization of Random Variables

- A continuous source contains an infinite amount of information and cannot be represented exactly using a finite number of bits.
- In lossy source coding, we seek instead a representation that is close to the source (with respect to some fidelity criterion) and can be represented using a finite number of bits.
- Quantization:
  - Let $X$ be a random variable
  - For every value $X = x$, we would like to find a representation $\hat{x}(x)$ where $\hat{x}$ can take on only $2^R$ different values for a given rate $R$ (measured in bits).

9.1.1 Example: Binary Source

- If we use 1 bit to represent $X$ (i.e., we can choose only two different reconstruction symbols), then we should use the bit to indicate whether $X$ is positive or negative.
- To minimize the square error distortion, the reconstruction symbols should be the conditional mean of its region,

$$\hat{x}(x) = \begin{cases} \mathbb{E}[X | X > 0], & x > 0 \\ \mathbb{E}[X | X < 0], & x < 0 \end{cases}$$

$$= \begin{cases} \sqrt{\frac{2}{\pi}} \sigma^2, & x > 0 \\ -\sqrt{\frac{2}{\pi}} \sigma^2, & x < 0 \end{cases}$$
The average distortion will then be
\[ E[(X - \hat{x}(X))^2] = \frac{1}{2} E[(X - \hat{x}(X))^2 | X > 0] + \frac{1}{2} E[(X - \hat{x}(X))^2 | X \leq 0] \]
\[ = E[(X - \hat{x}(X))^2 | X > 0] \]
\[ = \left(1 - \frac{2}{\pi}\right) \sigma^2 \]

If we are given 2 bits to represent \( X \), we should divide the real line into 4 regions and use the conditional mean within each region as the reconstruction point. However, it is not obvious how we should choose these regions.

- In general, a quantization scheme is characterized by a partition \( \{V_i\} \) of the space and the corresponding reconstruction points \( \{\hat{x}_i\} \).

\[ x \in V_i \implies \hat{x}(x) = \hat{x}_i \]

The regions and reconstruction points should satisfy:

- Given a set of reconstruction points, the regions should be chosen to minimized the distortion. This occurs if the regions are the Voronoi cells

\[ V_i = \{x : |x - x_i| < |x - x_j| \text{ for all } j \neq i\} \]

- Given a set of regions, the reconstruction points should be choses such that the distortion is minimized. Under squared error distortion, this is given by the conditional mean

\[ \hat{x}_i = E[X | X \in V_i] \]

**Lloyd’s Algorithm** is an iterative algorithm for constructing a quantization function. Starting with an initial set of reconstruction points, the algorithm repeats the following two steps:

1. Given reconstruction points, find optimal set of regions
2. Given regions, find optimal set of reconstruction points.

This algorithm will converge to a local optimum (but not necessarily the global optimum).

- **Vector Quantization**

- Let \( X^n = [X_1, \cdots, X_n] \) be an length-\( n \) random vector with iid entries

- For every realization \( X^n = x^n \) we would like to find a representation \( \hat{x}^n(x^n) \) where \( \hat{x}^n \) can take on only \( 2^{nR} \) different values for a given rate \( R \) (measured in bits).

- One option is to use the rate \( R \) scalar quantization strategy outlined above.

- It turns out that quantizing jointly can be much better than quantizing separately.

- **Recap:**

  - **scalar quantization**: Approximate \( X \) with \( R \) bits
  - **vector quantization**: Approximate \( X^n \) with \( nR \) bits
9.2 Lossy Source Coding Definitions

- Intuition: Using more bits reduces quantization error. How can we quantify this tradeoff?
- Illustration of a $(2^{nR}, n)$ lossy source code

\[
\begin{align*}
X^n & \xrightarrow{\text{source}} \text{encoder } f_n \xrightarrow{w \in \{1, 2, \cdots, 2^{nR}\}} \text{decoder } g_n \xrightarrow{\hat{X}^n} \text{reconstruction}
\end{align*}
\]

- The source produces a sequence $X_1, X_2, \cdots$ of iid random variable with distribution $p(x)$ supported on a finite alphabet $X$.
- The encoder is a mapping $f_n : X^n \rightarrow \{1, 2, \cdots, 2^{nR}\}$ that describes every source sequence by an index $w$. The rate is given by

\[
R = \frac{\log \# \text{ of indices}}{n} \text{ bits per symbol}
\]

- The decoder $g_n : \{1, 2, \cdots, 2^{nR}\} \rightarrow \hat{X}$ maps each index to an estimate $\hat{X} \in \hat{X}^n$ where $\hat{X}$ is the reconstruction alphabet.

- Definition: A per-letter distortion measure is a mapping

\[
d : X \times \hat{X} \rightarrow \mathbb{R}^+
\]

from the set of source alphabet-reconstruction pairs to the nonnegative real numbers. In most cases, the reconstruction alphabet is equal to the source alphabet. The distortion measure is bounded if the maximum value of the distortion is finite

\[
d \text{ is bounded } \iff \max_{x \in X, \hat{x} \in \hat{X}} d(x, \hat{x}) < \infty
\]

- The Hamming distortion is defined as

\[
d(x, \hat{x}) = \begin{cases} 
0, & \text{if } x = \hat{x} \\
1, & \text{if } x \neq \hat{x}
\end{cases}
\]

- The squared-error distortion is

\[
d(x, \hat{x}) = (x - \hat{x})^2
\]

- The distortion between two sequences $x^n$ and $\hat{x}^n$ is given by the average per-letter distortion

\[
d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i)
\]

- Definition: A $(2^{nR}, n)$ rate distortion coding scheme consists of

  - a source alphabet $X$ and reconstruction alphabet $\hat{X}$.
  - encoding function $f_n : X^n \rightarrow \{1, 2, \cdots, 2^{nR}\}$
  - decoding function $g_n : \{1, 2, \cdots, 2^{nR}\} \rightarrow \hat{X}^n$. 
a distortion measure \(d\)

- The distortion associate with the \((2^nR, n)\) coding scheme is given by the expected distortion
  \[
  D = E\left[d(X^n, \hat{X}^n)\right] = \sum_{x^n} p(x^n) d(x^n, g_n(f_n(x^n)))
  \]

- **Definition:** A rate distortion pair \((R, D)\) is said to be achievable for a source \(p(x)\) and distortion measure \(d(\cdot, \cdot)\) if there exists a sequence of \((2^nR, n)\) rate-distortion coding schemes with
  \[
  \lim_{n \to \infty} E[d(X^n, g_n(f_n(X^n)))] < D
  \]

- **Definition:** the rate distortion region for a source is the closure of the set of achievable distortion pairs \((R, D)\).

- **Definition:** The rate distortion function \(R(D)\) is the infimum of rates \(R\) such that \((R, D)\) is in the rate distortion region. Likewise, the distortion rate function \(D(R)\) is the infimum of distortions such that \((R, D)\) is in the rate distortion region.

### 9.3 The Rate Distortion Coding Theorem

- **Definition:** The information rate distortion function \(R^{(I)}(D)\) for a source \(p(x)\) with distortion measure \(d(x, \hat{x})\) is defined as
  \[
  R^{(I)}(D) = \min_{p(\hat{x}|x)} I(X; \hat{X})
  \]
  where the minimum is over all distributions \(p(\hat{x}|x)\) such that the pair \((X, \hat{X})\) satisfy the distortion constraint
  \[
  E\left[d(X, \hat{X})\right] \leq D, \quad (X, \hat{X}) \sim p(x)p(\hat{x}|x)
  \]

- **Theorem:** The rate distortion function for an i.i.d. source \(p(x)\) and bounded distortion measure \(d(\cdot, \cdot)\) is equal to the associated information rate distortion function, i.e.
  \[
  R(D) = R^{(I)}(D)
  \]
  \[
  \circ \text{ Achievability: } R > R^{(I)}(D) \implies \text{ the pair } (R, D) \text{ is achievable}
  \]
  \[
  \circ \text{ Converse: the pair } (R, D) \text{ is achievable } \implies R \geq R^{(I)}(D)
  \]

### 9.3.1 Example: Binary Source

- **Theorem:** The rate distortion function for a Bernoulli(\(p\)) source with Hamming distortion is given by
  \[
  R(D) = \begin{cases} 
  H(p) - H(D), & 0 \leq D \leq \min\{p, 1 - p\} \\
  0, & D > \min\{p, 1 - p\}
  \end{cases}
  \]

- **Proof of \(\geq\):
Need to show that for any distribution $p(\hat{x}|x)$,
\[ P\left[ X \neq \hat{X} \right] \leq D \implies I(X; \hat{X}) \geq R(D) \]

Without loss of generality, assume $p < 1/2$.

Note that $X \oplus \hat{X} = 1 \iff \hat{X} \neq X$

For any $p(\hat{x}|x)$ obeying $P\left[ X \neq \hat{X} \right] \leq D$, we have
\[
I(X; \hat{X}) = H(X) - H(X|\hat{X}) \\
= H(p) - H(X \oplus \hat{X}|\hat{X}) \\
\geq H(p) - H(X \oplus \hat{X}) \quad \text{Conditioning cannot increase entropy} \\
= H(p) - H_b\left( P\left[ X \neq \hat{X} \right] \right) \\
\geq H(p) - H_b(D) \quad \text{since } H(D) > H(D') \text{ for all } D' < D
\]

Thus, for $D < p$,
\[ R(D) \geq H(p) - H(D) \]

**Proof of $\leq$:**

The goal is to show that there exists $p(\hat{x}|x)$ such that $P\left[ X \neq \hat{X} \right] \leq D$ and $I(X; \hat{X}) \leq R(D)$

Without loss of generality, assume $p < 1/2$.

If $D \geq p$ then distortion is achieved automatically and $R = 0$ suffices.

Henceforth, we assume $D < p$.

The trick is to consider a channel with input $\hat{X}$ and output $X$ defined by $p(x|\hat{x})$. Note that given $p(x)$, there is a one-to-one mapping between $p(\hat{x}|x)$ and $p(x|\hat{x})$

For this problem, it is useful to consider the binary symmetric channel with cross over probability $D$ shown below:

We want choose the distribution on $\hat{X}$ so that $X$ is Bernoulli($p$). This require that
\[
P[X = 1] = P\left[ \hat{X} = 1 \right] (1 - D) + P\left[ \hat{X} = 0 \right] D = p
\]

or equivalently,
\[
P\left[ \hat{X} = 1 \right] = \frac{p - D}{1 - 2D}, \quad P\left[ \hat{X} = 0 \right] = \frac{1 - p - D}{1 - 2D}
\]

With this marginal distribution $\hat{X}$, it then follows that
\[ I(X; \hat{X}) = H(X) - H(X|\hat{X}) = H(p) - H(D) \]

and the expected distortion is $P\left[ X \neq \hat{X} \right] = D$. 
9.3.2 Example: Gaussian Source

The rate distortion coding theorem stated above assumes discrete sources and bounded distortion measures. It turns out that the theorem can also be proved for well-behaved continuous sources, such as for the case of Gaussian source with squared error distortion.

- **Theorem:** The rate distortion function for an $N(0, \sigma^2)$ source with squared-error distortion is

$$R(D) = \begin{cases} \frac{1}{2} \log \left( \frac{\sigma^2}{D} \right), & 0 \leq D \leq \sigma^2 \\ 0, & D > \sigma^2 \end{cases}$$

- **Proof of $\geq$:**

  - Let $(X, \hat{X})$ be distributed such that $X \sim N(0, \sigma^2)$ and $E[(X - \hat{X})^2] \leq D$.
  - The mutual information obeys the following inequalities:

    $$I(X; \hat{X}) = h(X) - h(X|\hat{X})$$
    $$= \frac{1}{2} \log (2\pi e\sigma^2) - h(X - \hat{X}|\hat{X})$$
    $$\geq \frac{1}{2} \log (2\pi e\sigma^2) - h(X - \hat{X})$$
    $$\geq \frac{1}{2} \log (2\pi e\sigma^2) - \frac{1}{2} \log \left(2\pi e E[(X - \hat{X})^2]\right)$$
    $$\geq \frac{1}{2} \log (2\pi e\sigma^2) - \frac{1}{2} \log (2\pi e D)$$
    $$= \frac{1}{2} \log \frac{\sigma^2}{D}$$

  - Hence, if $D < \sigma^2$,
  
    $$R(D) \geq \frac{1}{2} \log \frac{\sigma^2}{D}$$

- **Proof of $\leq$:**

  - Consider the inverse channel with input $\hat{X}$ and output $X$.
  - In this case, we choose

    $$X = \hat{X} + Z, \quad \hat{X} \sim N(0, \sigma^2 - D), \quad Z \sim N(0, D)$$

    where $\hat{X}$ and $Z$ are independent.
Then,
\[
I(X; \hat{X}) = h(X) - h(X|\hat{X}) \\
= \frac{1}{2} \log(2\pi e\sigma^2) - \frac{1}{2} \log(2\pi eD) \\
= \frac{1}{2} \log \frac{\sigma^2}{D}
\]
and also
\[
\mathbb{E}[(X - \hat{X})^2] = \mathbb{E}[Z^2] = D
\]

- The distortion rate function is given by
  \[
  D(R) = \sigma^2 2^{-2R}
  \]

- Comparison of scalar quantization and rate-distortion at rate \( R = 1 \)
  - Scalar quantization: \( D = (1 - \frac{2}{\pi})\sigma^2 \approx 0.36 \sigma^2 \)
  - Distortion rate function: \( D = \frac{1}{4} \sigma^2 \).

### 9.3.3 Properties of the Rate Distortion Functions

- **Theorem:** The rate distortion function \( R(D) \) is a non-increasing convex function of \( D \).

- **Proof:**
  - The rate distortion function is the minimum of the mutual information over increasingly larger sets as \( D \) increases. Thus \( R(D) \) is non-increasing.
  - Consider two rate distortion pairs \((R_1, D_1)\) and \((R_2, D_2)\) which lie on the rate distortion curve.
  - Let \( p_1(x, \hat{x}) = p(x)p_1(\hat{x}|x) \) and \( p_2(x, \hat{x}) = p(x)p_2(\hat{x}|x) \) be the distributions that achieve these pairs.
  - For \( \lambda \in [0, 1] \) consider the distribution
    \[
    p_\lambda(\hat{x}|x) = \lambda p_1(\hat{x}|x) + (1 - \lambda) p_2(\hat{x}|x)
    \]
  - By the linearity of expectation, the distortion associated with \( p_\lambda(x, \hat{x}) \) is given by
    \[
    D_\lambda = \lambda D_1 + (1 - \lambda) D_2
    \]
  - Since mutual information \( I(X; \hat{X}) \) is convex with respect to the conditional distribution \( p(x|\hat{x}) \), the rate distortion function at \( D_\lambda \) obeys
    \[
    R(D_\lambda) \leq I(p(x), p_\lambda(\hat{x}|x)) \\
    \leq \lambda I(p(x), p_1(\hat{x}|x)) + (1 - \lambda) I(p(x), p_2(\hat{x}|x)) \\
    = \lambda R(D_1) + (1 - \lambda) R(D_2)
    \]
9.4 Converse to the Rate Distortion Coding Theorem

- We now prove the converse to the rate distortion coding theorem. In particular, we show that for any \((2^nR, n)\) coding scheme,

\[
E[d(X^n, g_n(f_n(X^n)))] \leq D \implies R \geq R(D)
\]

- Let \(\hat{X}^n = g_n(f_n(X^n))\) and observe that:

\[
\begin{align*}
 nR & \geq H(f_n(X^n)) \\
 & \geq H(f_n(X^n)) - H(f_n(X^n)|X^n) \\
 & = I(X^n; f_n(X^n)) \\
 & \geq I(X^n; \hat{X}^n) \quad \text{data processing inequality} \\
 & = H(X^n) - H(X^n|\hat{X}^n) \\
 & = \sum_{i=1}^n H(X_i) - H(X^n|\hat{X}^n) \quad X_i \text{ are independent} \\
 & = \sum_{i=1}^n H(X_i) - \sum_{i=1}^n H(X_i|X^n, X_{i-1}, \ldots, X_1) \quad \text{chain rule} \\
 & \geq \sum_{i=1}^n H(X_i) - \sum_{i=1}^n H(X_i|\hat{X}_i) \quad \text{conditioning cannot increase entropy} \\
 & = \sum_{i=1}^n I(X_i; \hat{X}_i) \\
 & \geq \sum_{i=1}^n R\left(\mathbb{E}[d(X_i, \hat{X}_i)]\right) \quad \text{definition of rate distortion function} \\
 & = n\left(\frac{1}{n} \sum_{i=1}^n R\left(\mathbb{E}[d(X_i, \hat{X}_i)]\right)\right) \\
 & \geq nR\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[d(X_i, \hat{X}_i)]\right) \quad \text{convexity of } R(D) \text{ & Jensen’s inq.} \\
 & = nR\left(\mathbb{E}[d(X^n, \hat{X}^n)]\right) \quad \text{extension of distortion function} \\
 & \geq nR(D) \quad \text{since } R(D) \text{ is nonincreasing}
\end{align*}
\]
9.5 From Capacity to Rate Distortion

Throughout this section we use the logarithm of base two.

- Consider the AWGN channel: $Y = X + N$ with power constraint $P$ and noise power $N$.
- For each $n$ let $W$ be uniform on $\{1, 2, \ldots, 2^{nR}\}$.
- The capacity of the channel is $C = \frac{1}{2} \log_2(1 + \frac{P}{N})$ bits per channel use.
- Given $R < C$, consider a sequence of $(2^{nR}, n)$ coding schemes with $P_{e(n)} \to 0$ as $n \to \infty$.
- Then, it can be shown that
  \[
  \lim_{n \to \infty} \frac{1}{n} I(X^n; Y^n) = R
  \]
  where $X^n$ is the input to the channel defined by the coding scheme.

- Let $P_{Y^n}$ be the distribution of the output induced by the coding scheme. Let $Q_n$ be the i.i.d. Gaussian distribution with mean 0 and variance $V$. The relative entropy between these distributions is given by
  \[
  \frac{1}{n} D(P_{Y^n} \| Q_n) = \frac{1}{2} \log_2(2\pi e V) + \frac{1}{2} \frac{P + N}{V} - \frac{1}{n} h(Y^n) \\
  = \frac{1}{2} \log_2(2\pi e V) - \frac{1}{n} h(N^n) - \frac{1}{n} (h(Y^n) - h(N^n)) + \frac{1}{2} \frac{P + N - V}{V} \\
  = \frac{1}{2} \log_2 \left( \frac{V}{N} \right) - \frac{1}{n} I(X^n; Y^n) + \frac{1}{2} \frac{P + N - V}{V}
  \]
- Therefore,
  \[
  \lim_{n \to \infty} \frac{1}{n} D(P_{Y^n} \| Q_n) = \frac{1}{2} \log_2 \left( \frac{V}{N} \right) - R + \frac{1}{2} \frac{P + N - V}{V}
  \]
- Next, we apply Talagrand’s transportation inequality:
  \[
  W_2^2(P_{Y^n}, Q_n) \leq \frac{2V}{\log_2(e)} D(P_{Y^n} \| Q_n)
  \]
  and so
  \[
  \limsup_{n \to \infty} \frac{1}{n} W_2^2(P_{Y^n}, Q_n) \leq \frac{2V}{\log_2(e)} \left( \frac{1}{2} \log_2 \left( \frac{V}{N} \right) - R + \frac{1}{2} \frac{P + N - V}{V} \right)
  \]
- Let $V^*$ be the solution to
  \[
  \frac{1}{2} \log_2 \left( \frac{V}{N} \right) - R + \frac{1}{2} \frac{P + N - V}{V} = 0
  \]
  Then,
  \[
  \lim_{n \to \infty} \frac{1}{n} W_2^2(P_{Y^n}, Q_n^*) = 0
  \]
• Using the lemma, this means that
\[
\limsup_{n \to \infty} \frac{1}{n} W_2^2(P_{X^n}, Q_n^*) \leq N \tag{9.1}
\]

• What does this tell us? We have coding scheme with rate \( R \) that can approximate the Gaussian distribution of variance \( V^* \) with mean square distortion \( N \). How large is \( V^* \)? Observe that
\[
\frac{1}{2} \log \left( \frac{V^*}{N} \right) - \frac{1}{2} \frac{P + N - V^*}{V^*} = R = \frac{1}{2} \log \left( \frac{N + P}{N} \right) - \epsilon \tag{9.2}
\]
After staring at this for a bit we conclude that we have shown achievability of the rate distortion for some \( D \) such that \( \frac{1}{2} \log_2(D/\sigma^2) < R \).