ECE 587 / STA 563: Lecture 9 – Rate Distortion

Information Theory
Duke University, Fall 2023

Author: Galen Reeves
Last Modified: August 24, 2023

Outline of lecture:

9.1 Motivation: Quantization of Random Variables

- A continuous source contains an infinite amount of information and cannot be represented exactly using a finite number of bits.
- In lossy source coding, we seek instead a representation that is close to the source (with respect to some fidelity criterion) and can be represented using a finite number of bits.
- **Quantization:**
  - Let X be a random variable
  - For every value $X = x$, we would like to find a representation $\hat{x}(x)$ where $\hat{x}$ can take on only 2^R different values for a given rate R (measured in bits).
- **Example:** Quantizing a Gaussian random variable with squared error distortion:
  - Let $X \sim \mathcal{N}(0, \sigma^2)$
  - Consider mean squared error distortion
  $$\mathbb{E}\big[(X - \hat{x}(X))^2\big]$$
  - If we use 1 bit to represent X (i.e., we can chose only two different reconstruction symbols), then we should use the bit to indicate whether X is positive or negative.
  - To minimize the square error distortion, the reconstruction symbols should be the conditional mean given the sign
  $$\hat{x}(x) = \begin{cases} \mathbb{E}[X|X>0], & x>0 \\ \mathbb{E}[X|X<0], & x<0 \end{cases} = \begin{cases} \sqrt{\frac{2}{\pi}\sigma^2}, & x>0 \\ -\sqrt{\frac{2}{\pi}\sigma^2}, & x<0 \end{cases}$$

- The average distortion will then be

$$\begin{aligned}\mathbb{E}\big[(X-\hat{x}(X))^2\big] &= \frac{1}{2}\mathbb{E}\big[(X-\hat{x}(X))^2|X>0\big]+\frac{1}{2}\mathbb{E}\big[(X-\hat{x}(X))^2|X\le 0\big]\\ &= \mathbb{E}\big[(X-\hat{x}(X))^2|X>0\big]\\ &= \left(1-\frac{2}{\pi}\right)\sigma^2\end{aligned}$$

- If we are given 2 bits to represent X, we should divide the real line into 4 regions and use the conditional means within each region as the reconstruction points. However, it is not obvious how we should choose these regions.

- In general, a quantization scheme is characterized by a partition $\{V_i\}$ of the space and the corresponding reconstruction points $\{\hat{x}_i\}$.

$$x \in V_i \quad \implies \hat{x}(x) = \hat{x}_i$$

The regions and reconstruction points should satisfy:

- Given a set of reconstruction points, the regions should be chosen to minimize the distortion. Under the squared-error distortion, this occurs if the regions are the Voronoi cells
$$V_i = \{x : \|x - x_i\| < \|x - x_j\| \text{ for all } j \neq i\}$$

- Given a set of regions, the reconstruction points should be chosen to minimize the distortion. Under squared error distortion, this is given by the conditional mean

$$\hat{x}_i = \mathbb{E}[X|X \in V_i]$$

- **Lloyd's Algorithm** is an iterative algorithm for constructing a quantization function. Starting with an initial set of reconstruction points, the algorithm repeats the following two steps:

(1) Given reconstruction points, find optimal set of regions

(2) Given regions, find optimal set of reconstruction points.

This algorithm will converge to a local optimum (but not necessarily the global optimum).

- **Vector Quantization**

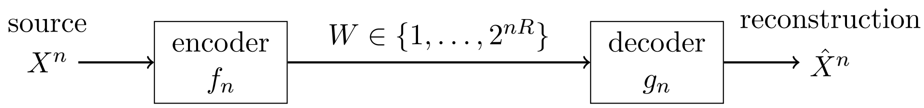
  - Let $X^n = [X_1, \cdots, X_n]$ be an length-n random vector with iid entries
  - For every realization $X^n = x^n$ we would like to find a representation $\hat{x}^n(x^n)$ where $\hat{x}^n$ can take on only 2^{nR} different values for a given rate R (measured in bits).
  - One option is to use the rate R scalar quantization strategy outlined above.
  - It turns out that quantizing jointly can be *much better* than quantizing separately.

- Recap:

  - **scalar quantization:** Approximate X with R bits
  - **vector quantization:** Approximate X^n with nR bits

9.2 Lossy Source Coding Definitions

- Intuition: Using more bits reduces quanitization error. How can we quantify this this tradeoff?
- Illustration of a $(2^{nR}, n)$ lossy source code



- The **source** produces a sequence $X_1, X_2, \cdots$ of iid random variable with distribution $p(x)$ supported on a finite alphabet $\mathcal{X}$.
- The **encoder** is a mapping $f_n : \mathcal{X}^n \to \{1, 2, \cdots, 2^{nR}\}$ that describes every source sequence by an index w. The rate is given by

$$R = \frac{\log \# \text{ of indices}}{n} \quad \text{bits per symbol}$$

- The **decoder** $g_n : \{1, 2, \cdots, 2^{nR}\} \to \hat{X}$ maps each index to an estimate $\hat{X} \in \hat{\mathcal{X}}^n$ where $\hat{\mathcal{X}}$ is the reconstruction alphabet.
- **Definition:** A *per-letter distortion measure* is a mapping

$$d : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}^+$$

from the set of source alphabet - reconstruction pairs to the nonnegative real numbers. In most cases, the reconstruction alphabet is equal to the source alphabet. The distortion measure is *bounded* it the maximum value of the distortion is finite

$$d \text{ is bounded} \quad \iff \quad \max_{x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}} d(x, \hat{x}) < \infty$$

- The **Hamming distortion** is defined as

$$d(x, \hat{x}) = \begin{cases} 0, & \text{if } x = \hat{x} \\ 1, & \text{if } x \neq \hat{x} \end{cases}$$

- The **squared-error distortion** is defined as

$$d(x, \hat{x}) = (x - \hat{x})^2$$

- The distortion between two sequences x^n and $\hat{x}^n$ is given by the average per-letter distortion

$$d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i)$$

- **Definition:** A $(2^{nR}, n)$ rate distortion coding scheme consists of
    - a source alphabet $\mathcal{X}$ and reconstruction alphabet $\hat{\mathcal{X}}$.
    - encoding function $f_n : X^n \to \{1, 2, \cdots, 2^{nR}\}$
    - decoding function $g_n : \{1, 2, \cdots, 2^{nR}\} \to \hat{X}^n$.

- a distortion measure d

- The distortion associate with the $(2^{nR}, n)$ coding scheme is given by the expected distortion

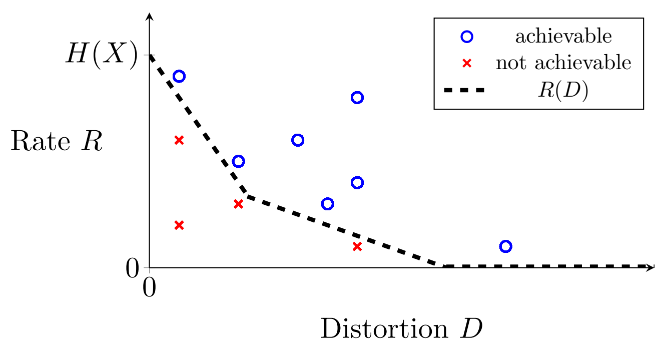
$$\mathbb{E}\Big[d(X^n, \hat{X}^n)\Big] = \sum_{x^n} p(x^n) d(x^n, g_n(f_n(x^n)))$$

- **Definition:** A rate distortion pair (R, D) is said to be *achievable* for a source $p(x)$ and distortion measure $d(\cdot,\cdot)$ if there exists a sequence of $(2^{nR}, n)$ rate-distortion coding schemes with

$$\limsup_{n\to\infty} \mathbb{E}[d(X^n, g_n(f_n(X^n)))] \le D$$

- **Definition:** the *rate distortion region* for a source is the closure of the set of achievable distortion pairs (R, D).

- **Definition:** The *rate distortion function* $R(D)$ is the infimum of rates R such that (R, D) is in the rate distortion region. Likewise, the *distortion rate function* $D(R)$ is the infimum of distortions such that (R, D) is in the rate distortion region.



9.3 The Rate Distortion Coding Theorem

- **Definition:** The *information rate distortion function* $R^{(I)}(D)$ for a source $p(x)$ with distortion measure $d(x, \hat{x})$ is defined as

$$R^{(I)}(D) = \min_{p(\hat{x}|x)} I(X; \hat{X})$$

where the minimum is over all distributions $p(\hat{x}|x)$ such that the pair $(X, \hat{X})$ satisfy the distortion constraint

$$\mathbb{E}\Big[d(X, \hat{X})\Big] \le D, \qquad (X, \hat{X}) \sim p(x)p(\hat{x}|x)$$

- **Theorem:** The rate distortion function for an i.i.d. source $p(x)$ and bounded distortion measure $d(\cdot,\cdot)$ is equal to the associated information rate distortion function, i.e.

$$R(D) = R^{(I)}(D)$$

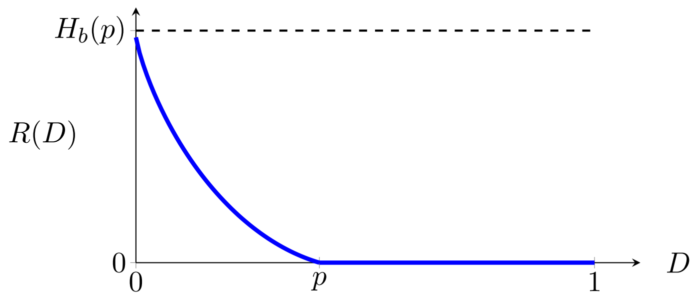
  - Achievability: $R > R^{(I)}(D) \implies$ the pair (R, D) is achievable

  - Converse: the pair (R, D) is achievable $\implies R \ge R^{(I)}(D)$

9.3.1 Example: Binary Source

- **Theorem**: The rate distortion function for an iid Bernoulli(p) source with Hamming distortion is given by

$$R(D) = \begin{cases} H(p) - H(D), & 0 \le D \le \min\{p, 1-p\} \\ 0, & D > \min\{p, 1-p\} \end{cases}$$



- **Proof of $\ge$**:

  - Need to show that for any distribution $p(\hat{x}|x)$,

$$\mathbb{P}\left[X \neq \hat{X}\right] \le D \implies I(X;\hat{X}) \ge R(D)$$

  - Without loss of generality, assume $p < 1/2$.
  - Note that $X \oplus \hat{X} = 1 \Longleftrightarrow \hat{X} \neq X$
  - For any $p(\hat{x}|x)$ obeying $\mathbb{P}\left[X \neq \hat{X}\right] \le D$, we have

$$\begin{aligned} I(X;\hat{X}) &= H(X) - H(X|\hat{X}) \\ &= H(p) - H(X \oplus \hat{X}|\hat{X}) \\ &\ge H(p) - H(X \oplus \hat{X}) && \text{Conditioning cannot increase entropy} \\ &= H(p) - H_b\Big(\mathbb{P}\left[X \neq \hat{X}\right]\Big) \\ &\ge H(p) - H_b(D) && \text{since } H(D) > H(D') \text{ for all } D' < D \end{aligned}$$

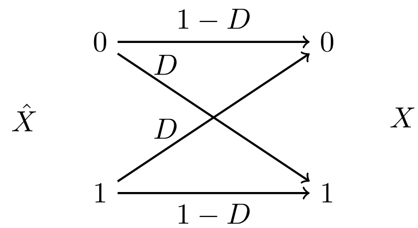
  - Thus, for $D < p$,

$$R(D) \ge H(p) - H(D)$$

- **Proof of $\le$**:

  - The goal is to show that there exists $p(\hat{x}|x)$ such that $\mathbb{P}\left[X \neq \hat{X}\right] \le D$ and $I(X;\hat{X}) \le R(D)$
  - Without loss of generality, assume $p < 1/2$.
  - If $D \ge p$ then distortion is achieved automatically and $R = 0$ suffices.
  - Henceforth, we assume $D < p$.
  - The trick is to consider a channel with input $\hat{X}$ and output X defined by $p(x|\hat{x})$. Note that given $p(x)$, there is a one-to-one mapping between $p(\hat{x}|x)$ and $p(x|\hat{x})$

- For this problem, it is useful to consider the binary symmetric channel with cross over probability D shown below:



- We want choose the distribution on $\hat{X}$ so that X is Bernoulli(p). This requires that

$$\mathbb{P}[X=1] = \mathbb{P}\left[\hat{X}=1\right](1-D) + \mathbb{P}\left[\hat{X}=0\right]D = p$$

  or equivalently,

$$\mathbb{P}\left[\hat{X}=1\right] = \frac{p-D}{1-2D}, \qquad \mathbb{P}\left[\hat{X}=0\right] = \frac{1-p-D}{1-2D}$$

- With this marginal distribution $\hat{X}$, it then follows that

$$I(X;\hat{X}) = H(X) - H(X|\hat{X}) = H(p) - H(D)$$

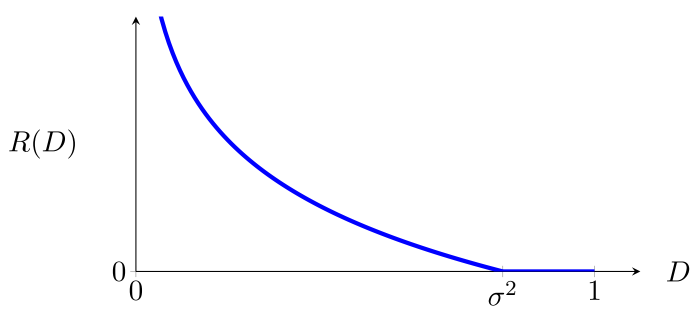
  and the expected distortion is $\mathbb{P}\left[X \neq \hat{X}\right] = D$.

9.3.2 Example: Gaussian Source

The rate distortion coding theorem stated above assumes discrete sources and bounded distortion measures. It turns out that the theorem can also be proved for well-behaved continuous sources, such as for the case of Gaussian source with squared error distortion.

- **Theorem:** The rate distortion function for an $\mathcal{N}(0,\sigma^2)$ source with squared-error distortion is

$$R(D) = \begin{cases} \frac{1}{2}\log\left(\frac{\sigma^2}{D}\right), & 0 \le D \le \sigma^2 \\ 0, & D > \sigma^2 \end{cases}$$



- **Proof of $\geq$:**
  - Let $(X,\hat{X})$ be distributed such that $X \sim N(0,\sigma^2)$ and $\mathbb{E}\left[(X-\hat{X})^2\right] \le D$.

- The mutual information obeys the following inequalities:

$$\begin{aligned} I(X;\hat{X}) &= h(X) - h(X|\hat{X}) \\ &= \frac{1}{2}\log\big(2\pi e\sigma^2\big) - h(X-\hat{X}|\hat{X}) \\ &\geq \frac{1}{2}\log\big(2\pi e\sigma^2\big) - h(X-\hat{X}) \\ &\geq \frac{1}{2}\log\big(2\pi e\sigma^2\big) - \frac{1}{2}\log\Big(2\pi e\mathbb{E}\Big[(X-\hat{X})^2\Big]\big)\Big) \\ &\geq \frac{1}{2}\log\big(2\pi e\sigma^2\big) - \frac{1}{2}\log(2\pi eD) \\ &= \frac{1}{2}\log\frac{\sigma^2}{D} \end{aligned}$$

- Hence, if $D < \sigma^2$,

$$R(D) \geq \frac{1}{2}\log\frac{\sigma^2}{D}$$

- **Proof of $\leq$:**
  - Consider the inverse channel with input $\hat{X}$ and output X.
  - In this case, we choose the Gaussian channel:

$$X = \hat{X} + Z, \qquad \hat{X} \sim \mathcal{N}(0, \sigma^2 - D), \qquad Z \sim \mathcal{N}(0, D)$$

    where $\hat{X}$ and Z are independent.
  - Then,

$$\begin{aligned} I(X;\hat{X}) &= h(X) - h(X|\hat{X}) \\ &= \frac{1}{2}\log(2\pi e\sigma^2) - \frac{1}{2}\log(2\pi eD) \\ &= \frac{1}{2}\log\frac{\sigma^2}{D} \end{aligned}$$

    and also

$$\mathbb{E}\Big[(X-\hat{X})^2\Big] = \mathbb{E}\big[Z^2\big] = D$$

- The distortion rate function is given by

$$D(R) = \sigma^2 2^{-2R}$$

- Comparison of scalar quantization and rate-distortion at rate $R = 1$
  - Scalar quantization: $D = (1 - \frac{2}{\pi})\sigma^2 \approx 0.36\,\sigma^2$
  - Distortion rate function: $D = \frac{1}{4}\sigma^2$.

9.3.3 Properties of the Rate Distortion Functions

- **Theorem:** The rate distortion function $R(D)$ is a non-increasing convex function of D.
- **Proof:**

- The rate distortion function is the minimum of the mutual information over increasingly larger sets as D increases. Thus $R(D)$ is non-increasing.
- Consider two rate distortion pairs (R_1, D_1) and (R_2, D_2) which lie on the rate distortion curve.
- Let $p_1(x, \hat{x}) = p(x)p_1(\hat{x}|x)$ and $p_2(x, \hat{x}) = p(x)p_2(\hat{x}|x)$ be the distributions that achieve these pairs.
- For $\lambda \in [0, 1]$ consider the distribution

$$p_\lambda(\hat{x}|x) = \lambda\, p_1(\hat{x}|x) + (1 - \lambda)\, p_2(\hat{x}|x)$$

- By the linearity of expectation, the distortion associated with $p_\lambda(x, \hat{x})$ is given by

$$D_\lambda = \lambda D_1 + (1 - \lambda) D_2$$

- Since mutual information $I(X; \hat{X})$ is convex with respect to the conditional distribution $p(x|\hat{x})$, the rate distortion function at D_λ obeys

$$\begin{aligned} R(D_\lambda) &\le I(p(x), p_\lambda(\hat{x}|x)) \\ &\le \lambda\, I(p(x), p_1(\hat{x}|x)) + (1 - \lambda)\, I(p(x), p_2(\hat{x}|x)) \\ &= \lambda\, R(D_1) + (1 - \lambda)\, R(D_2) \end{aligned}$$

9.4 Converse to the Rate Distortion Coding Theorem

- We now prove the converse to the rate distortion coding theorem. In particular, we show that for any $(2^{nR}, n)$ coding scheme,

$$\mathbb{E}[d(X^n, g_n(f_n(X^n)))] \le D \quad \Longrightarrow \quad R \ge R(D)$$

- Let $\hat{X}^n = g_n(f_n(X^n))$ and observe that:

$$\begin{aligned} nR &\ge H(f_n(X^n)) && \text{range of } f_n \text{ at most } 2^{nR} \\ &\ge H(f_n(X^n)) - H(f_n(X^n)|X^n) \\ &= I(X^n; f_n(X^n)) \\ &\ge I(X^n; \hat{X}^n) && \text{data processing inequality} \\ &= H(X^n) - H(X^n|\hat{X}^n) \\ &= \sum_{i=1}^n H(X_i) - H(X^n|\hat{X}^n) && X_i \text{ are independent} \\ &= \sum_{i=1}^n H(X_i) - \sum_{i=1}^n H(X_i|\hat{X}^n, X_{i-1}, \cdots, X_1) && \text{chain rule} \\ &\ge \sum_{i=1}^n H(X_i) - \sum_{i=1}^n H(X_i|\hat{X}_i) && \text{conditioning cannot increase entropy} \\ &= \sum_{i=1}^n I(X_i; \hat{X}_i) \\ &\ge \sum_{i=1}^n R\Big(\mathbb{E}\Big[d(X_i, \hat{X}_i)\Big]\Big) && \text{definition of rate distortion function} \end{aligned}$$

$$
\begin{aligned}
&= n\left(\frac{1}{n}\sum_{i=1}^{n} R\Big(\mathbb{E}\Big[d(X_i,\hat{X}_i)\Big]\Big)\right) \\
&\geq n\,R\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\Big[d(X_i,\hat{X}_i)\Big]\right) && \text{convexity of } R(D) \text{ \& Jensen's inq.} \\
&= n\,R\Big(\mathbb{E}\Big[d(X^n,\hat{X}^n)\Big]\Big) && \text{extension of distortion function} \\
&\geq nR(D) && \text{since } R(D) \text{ is nonincreasing}
\end{aligned}
$$