ECE 587 / STA 563: Lecture 8 – Gaussian Channel

Information Theory
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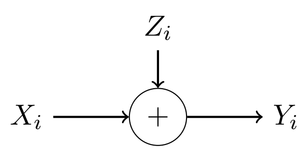
Outline of lecture:

8.1 Gaussian channel

- In many real-world applications, the difference between what is sent X and what is received Y can be modeled as additive white Gaussian noise.

- The discrete time Gaussian channel is given by



  where $Z_i \sim N(0, N)$ is independent of X_i.

- Without any constraints, the capacity is infinite!

- To mode real-world constraint, impose *average power constraint* on codewords $(x_1, x_2, \cdots, x_n)$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P$$

- **Example:** A simple strategy for communication on the AWGN channel

  - Send $X = +\sqrt{P}$ to communicate 1 and $-\sqrt{P}$ to communicate 0. This obeys the average power constraint.
  - The received signal is
  $$Y = \pm\sqrt{P} + Z$$
  - Since noise is symmetric, the optimal decoder is given by
  $$\begin{aligned} &\text{If } Y \geq 0 \implies \text{decide } +\sqrt{P} \\ &\text{If } Y < 0 \implies \text{decide } -\sqrt{P} \end{aligned}$$

- The probability of error is

$$\begin{aligned} P_e &= \mathbb{P}\Big[\text{error } | X = \sqrt{P}\Big]\frac{1}{2} + \mathbb{P}\Big[\text{error } | X = -\sqrt{P}\Big]\frac{1}{2} \\ &= \frac{1}{2}\mathbb{P}\Big[Z < -\sqrt{P}\Big] + \frac{1}{2}\mathbb{P}\Big[Z \geq \sqrt{P}\Big] \\ &= \mathbb{P}\Big[Z > \sqrt{P}\Big] \\ &= \mathbb{P}\Big[N(0,1) > \sqrt{P/N}\Big] \\ &= 1 - \Phi(\sqrt{P/N}) \end{aligned}$$

  where $\Phi(x)$ is the CDF of the standard Gaussian distribution

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

- In this example, we have converted Gaussian channel into a discrete BSC with $p = P_e$. It turns out we can do much better (at least when when the SNR P/N is large).

- **Definition:** The *information capacity* of the Gaussian channel is

$$C = \max_{f(x)\,:\,\mathbb{E}[X^2]\leq P} I(X;Y)$$

- **Theorem:** The information capacity of the Gaussian channel with additive noise power N and power constraint P is

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

- Proof:
  - The mutual information can be expressed as

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) \\ &= h(Y) - h(X+Z|X) \\ &= h(Y) - h(Z) \\ &= h(Y) - \frac{1}{2}\log(2\pi e N) \end{aligned}$$

  - The maximum of the first term occurs when Y is Gaussian:

$$\begin{aligned} \max_{f(x)\,:\,\mathbb{E}[X^2]\leq P} h(Y) &= \max_{f(x)\,:\,\mathbb{E}[X^2]\leq P} h(X+Z) \\ &\leq \max_{f(y)\,:\,\mathbb{E}[Y^2]\leq P+N} h(Y) \\ &= \frac{1}{2}\log(2\pi e(N+P)) \end{aligned}$$

  - Putting everything together gives

$$I(X;Y) \leq \frac{1}{2}\log\left(\frac{N+P}{N}\right) = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

  - This holds with equality when $X \sim N(0, P)$

- **Definition:** A rate R is *achievable* for the Gaussian channel with power constraint P if there exists a sequence of $(2^{nR}, n)$ coding schemes satisfying the power constraint such that the maximal probability of error converges to zero as n becomes large. The *capacity* of the channel is the supremum of the achievable rates.

- **Theorem:** The capacity of the Gaussian channel with additive noise power N and power constraint P is equal to the information capacity:

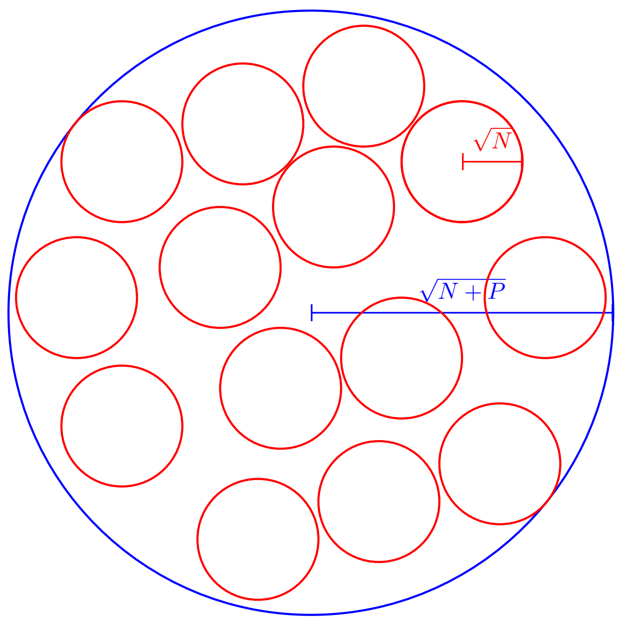
$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right) \qquad \text{bits per transmission}$$

- Proof:

  (1) Achievability: $R < C \implies R$ is achievable.

  (2) Converse: R is achievable $\implies R < C$.

- For an intuitive explanation of the channel capacity, consider *sphere packing.*



  - Suppose codeword $x^n(i)$ is sent. The received vector $Y^n = x^n(i) + Z^n$ obeys

$$\frac{1}{n}\mathbb{E}\big[\|Y^n - x^n(i)\|^2\big] = \frac{1}{n}\mathbb{E}\big[\|Z^n\|^2\big] = N$$

    Thus, with high probability, the received vector is contained in a sphere is radius $\sqrt{n(N+\epsilon)}$ around the true codeword.

$$\frac{1}{\sqrt{n}}\|Y^n - x^n(i)\| \leq \sqrt{N+\epsilon}$$

  - If our decoder assigns everything in this region to the i-th message, then an error occurs under the following events:

    (1) the received error falls out side the ball (i.e., big noise);

    (2) the sphere of another codeword $x^n(j)$ overlaps with the i-th sphere.

- Since the codewords obey an average power constraint, the average power of Y^n obeys

$$\frac{1}{n}\mathbb{E}\big[\|Y^n\|^2\big] = \frac{1}{n}\mathbb{E}\big[\|X^n(W) + Z^n\|^2\big] \le P + N$$

  and thus, with high probability,

$$\frac{1}{\sqrt{n}}\|Y^n\| \le \sqrt{P + N + \epsilon}$$

- The number M of messages we can send reliably is given by the number of spheres of radius $\approx \sqrt{N}$ that can we pack inside sphere of radius $\approx \sqrt{P+N}$
- The volume of an n-dimensional sphere of radius r is

$$\frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n = C_n r^n$$

- Thus, the maximum number of nonintersecting spheres is upper bounded by the ratio of the volumes

$$M \le \frac{\text{volume large sphere}}{\text{volume of small sphere}} = \frac{C_n (N+P)^{n/2}}{C_n (N)^{n/2}} = \left(1 + \frac{P}{N}\right)^{n/2}$$

- The rate of the corresponding code is

$$R = \frac{\log M}{n} = \frac{1}{n}\frac{n}{2}\log\left(1 + \frac{P}{N}\right) = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

8.1.1 Parallel Gaussian Channels

- Consider k independent Gaussian channels in parallel with a common power constraint.

$$\begin{aligned} Y_1 &= X_1 + Z_1 \\ Y_2 &= X_2 + Z_2 \\ &\vdots \\ Y_k &= X_k + Z_k \end{aligned}$$

- **Examples:**
  - OFDM (orthogonal frequency-division multiplexing), parallel channels formed in frequency domain
  - MIMO (multiple-input-multiple-output) - multiple antenna systems
- The noise Z_i is independent with

$$Z_i \sim N(0, N_i)$$

- The total power constraint across the channels

$$\sum_{j=1}^{k} P_i \le P, \qquad P_i = \mathbb{E}\big[X_i^2\big]$$

- The goal is to distribute the power amongst the channels to maximize the total capacity

- For any allocation of powers obeying the power constraint, the mutual information obeys

$$\begin{aligned}
I(X^k;Y^k) &= h(Y^k) - h(Y^k|X^k) \\
&= h(Y^k) - h(Z^k|X^k) && \text{(shift invariance)} \\
&= h(Y^k) - h(Z^k) && \text{(Z^k independent of X^k)} \\
&= h(Y^k) - \sum_{i=1}^k \frac{1}{2}\log(2\pi e N_i) \\
&\le \sum_{i=1}^k h(Y_i) - \sum_{i=1}^k \frac{1}{2}\log(2\pi e N_i) && \text{(independence bound)} \\
&\le \sum_{i=1}^k \frac{1}{2}\log(2\pi e (P_i + N_i)) - \sum_{i=1}^k \frac{1}{2}\log(2\pi e N_i) && \text{(Gaussian has max entropy)} \\
&= \sum_{i=1}^k \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right)
\end{aligned}$$

- This upper bound is achieved when X_i are independent with

$$X_i \sim N(0, P_i)$$

and thus the capacity is

$$\begin{aligned}
C &= \max_{X^k \,:\, \sum_i \mathbb{E}[X_i^2] \le P} I(X^k;Y^k) \\
&= \max_{P_i} \sum_{i=1}^k \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right), \qquad \text{subject to} \quad \sum_i P_i \le P, \quad P_i \ge 0
\end{aligned}$$

- This is constrained optimization problem. Consider the Lagrangian

$$J(P_1, \cdots, P_k, \lambda) = \sum_{i=1}^k \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right) - \lambda\left(\sum_i P_i - P\right)$$

- Since

$$\inf_{\lambda > 0}\left(\sum_{i=1}^k \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right) - \lambda\left(\sum_i P_i - P\right)\right) = \begin{cases} \sum_{i=1}^k \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right), & \sum_i P_i \le P \\ -\infty, & \sum_i P_i > P \end{cases}$$

the objective function can be expressed as

$$C = \sup_{P_i \ge 0} \inf_{\lambda \ge 0} J(P_1, \cdots, P_k, \lambda)$$

- Swapping the order of the max and sup provides an *upper bound*:

$$C \le \inf_{\lambda \ge 0} \sup_{P_i \ge 0} J(P_1, \cdots, P_k, \lambda)$$

- For each λ, the derivative of $J(P_1, \cdots, P_k, \lambda)$ with respect to P_i is given by

$$\frac{\partial}{\partial P_i} J(P^k, \lambda) = \frac{1}{2(N_i + P_i)} - \lambda$$

thus

$$P_i < \frac{1}{2\lambda} - N_i \quad \Longrightarrow \quad J(P_1, \cdots, P_k, \lambda) \text{ is increasing with respect to } P_i$$
$$P_i > \frac{1}{2\lambda} - N_i \quad \Longrightarrow \quad J(P_1, \cdots, P_k, \lambda) \text{ is decreasing with respect to } P_i$$

Recalling that P_i are nonnegative, it thus follows that $J(P_1, \cdots, P_k, \lambda)$ is maximized when

$$P_i^*(\lambda) = \begin{cases} \frac{1}{2\lambda} - N_i, & \frac{1}{2\lambda} - N_i > 0 \\ 0, & \frac{1}{2\lambda} - N_i \leq 0 \end{cases} = \max\left(\frac{1}{2\lambda} - N_i, 0\right)$$

and thus, for each $\lambda > 0$,

$$\max_{P_i \geq 0} J(P_1, \ldots, P_k, \lambda) = \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{P_i^*(\lambda)}{N_i}\right) - \lambda\left(\sum_i P_i^*(\lambda) - P\right)$$

- The above expression provides an upper bound on C. Let λ^* be the value of λ such the power constraint is met with equality, i.e.

$$\sum_i P_i^*(\lambda^*) = P$$

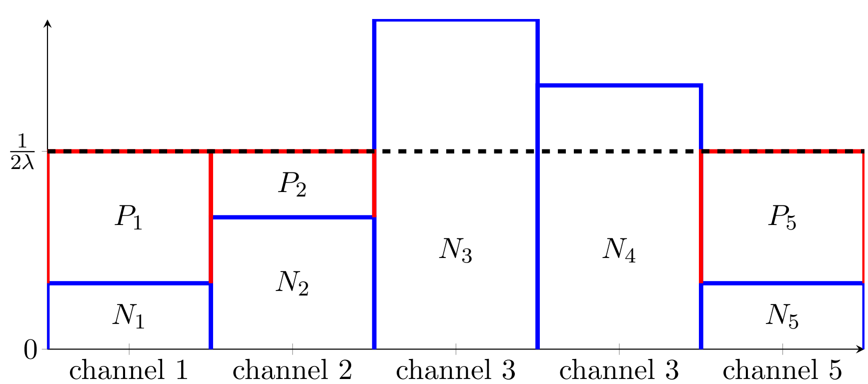
This value is guaranteed to exists because $P_i^*(\lambda)$ is continuous and increasing in λ. Then,

$$C \leq J(P_1^*(\lambda^*), \ldots, P_k^*(\lambda^*), \lambda^*) = \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{P_i^*(\lambda^*)}{N_i}\right)$$

- By construction, the power allocation $P_i^*(\lambda^*)$ satisfies the constraints of our problem and thus is a feasible solution. Since it matches the upper bound on the objective function, we know that it is the optimal solution. We conclude that

$$C = \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{P_i^*(\lambda^*)}{N_i}\right)$$

- This solution is known as *water filling*.



- This same story also holds true if the noise is correlated between channels. We apply water filling to the the *eigenvalues* of the covariance matrix K of the noise.