7.1 Entropy of Continuous Variables

Let $X$ be a continuous real-valued random variable with probability density function (pdf) $f_X(x)$ given by:

$$
P[X \leq x] = \int_{-\infty}^{x} f_X(t) dt$$

- Divide range of $X$ into bins of length $\Delta$.

- By mean value theorem, there exists a valued $x_i$ in the $i$th bin such that

$$f(x_i)\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x) dx$$

- Consider the quantized random variable $X^\Delta$ defined by

$$X^\Delta = x_i \text{ if } i\Delta \leq X < (i + 1)\Delta$$

- The random variable $X^\Delta$ has alphabet \{x_1, x_2, \cdots \} and pmf

$$p_{X^\Delta}(x_i) = f(x_i)\Delta$$
• The entropy of the quantized variable $X^\Delta$ is

$$H(X^\Delta) = -\sum_i p(x_i) \log p(x_i)$$

$$= -\sum_i \Delta f(x_i) \log(f(x_i)\Delta)$$

$$= -\sum_i \Delta f(x_i) \log f(x_i) - \sum_i \Delta f(x_i) \log \Delta$$

$$= -\sum_i \Delta f(x_i) \log f(x_i) - \log \Delta$$

• If the function $f_X(x) \log f_X(x)$ is Riemann integrable, then the limit of the first term as $\Delta$ becomes small is given by

$$\sum_i \Delta f_X(x_i) \log f_X(x_i) \to \int f_X(x) \log f_X(x) dx, \quad \text{as } \Delta \to 0$$

• Thus, for small $\Delta$, we have

$$H(X^\Delta) \approx \int f_X(x) \log \left(\frac{1}{f_X(x)}\right) dx + \log \left(\frac{1}{\Delta}\right)$$

• Therefore:

1. As $\Delta \to 0$, the entropy of the quantized version blows up

$$H(X^\Delta) \to \infty \quad \text{as } \Delta \to 0$$

This means the entropy of a continuous random variable is infinite.

2. As $\Delta \to 0$, the difference between the entropy of the quantized version and $\log(1/\Delta)$ satisfies

$$\lim_{\Delta \to 0} \left( H(X^\Delta) - \log \left(\frac{1}{\Delta}\right) \right) = \int f_X(x) \log \left(\frac{1}{f_X(x)}\right) dx$$

### 7.2 Differential Entropy

• **Definition:** The differential entropy $h(X)$ of a continuous random variable $X$ is

$$h(X) = -\int f(x) \log f(x) dx$$

Sometimes denoted $h(f)$.

• **Example:** Uniform distribution:

  o The pdf is given by

  $$f(x) = 1/a, \quad x \in [0, a]$$

  o The differential entropy is $h(X) = \int_0^a \frac{1}{a} \log(a) dx = \log a$

  o Note that for $a < 1$, we have $\log a < 0$ and so differential entropy can be negative!

  o Note that $2^{h(X)} = 2^{\log a} = a$ is the size of the support set.
• **Example:** Normal distribution
  
  ○ The pdf is given by
  \[
  f(x) = \phi(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{x^2}{2\sigma^2}}
  \]
  
  ○ The differential entropy measured in nats is
  \[
  h(\phi) = \int_{-\infty}^{\infty} \phi(x) \ln \phi(x) dx
  = \mathbb{E} \ln \phi(X)
  = \mathbb{E} \left[ \frac{X^2}{2\sigma^2} + \frac{1}{2} \ln 2\pi \sigma^2 \right]
  = \frac{1}{2} \ln e + \frac{1}{2} \ln (2\pi \sigma^2)
  = \frac{1}{2} \ln 2\pi e \sigma^2,
  \text{nats}
  \]
  
  ○ changing the base gives
  \[
  h(\phi) = \frac{1}{2} \log 2\pi e \sigma^2
  \text{ bits}
  \]
  
  ○ for \( a < 1 \), we have \( \log a < 0 \) and so differential entropy can be negative!
  
  ○ note that \( 2^{h(X)} = 2^{\log a} = a \) is the size of the support set.

• The **joint differential entropy** between \( X \) and \( Y \) is defined by
  \[
  h(X, Y) = \int f_{X,Y}(x, y) \log \left( \frac{1}{f_{X,Y}(x, y)} \right) dx dy
  \]

• The **conditional differential entropy** of \( X \) given \( Y \) is defined by
  \[
  h(X \mid Y) = -\int f(x, y) \log f(x \mid y) dx dy
  \]
  
  It can also be expressed as
  \[
  h(X\mid Y) = h(X, Y) - h(Y)
  \]

• The **Relative entropy** between densities \( f \) and \( g \) is
  \[
  D(f \| g) = \int f(x) \log \frac{f(x)}{g(x)} dx
  \]

• The **mutual information** between \( X \) and \( Y \) is
  \[
  I(X; Y) = \int f(x, y) \log \frac{f(x, y)}{f(x)f(y)} dx dy
  \]

• Note that
  \[
  I(X; Y) = h(X) - h(X \mid Y)
  = h(Y) - h(Y \mid X)
  = D(f(x, y)\|f(x)f(y))
  \]
- Venn diagram of relationship between mutual information and differential entropy.

![Venn Diagram](image)

- **Example:** (Bivariate Gaussian Distribution) Let \((X, Y) \sim N(0, K)\) be jointly Gaussian with mean zero and covariance \(K\) given by

\[
K = \begin{bmatrix}
\sigma^2 & \rho \sigma^2 \\
\rho \sigma^2 & \sigma^2
\end{bmatrix}
\]

- From the previous example, we know that

\[
h(X) = \frac{1}{2} \log(2\pi e \sigma^2), \quad h(Y) = \frac{1}{2} \log(2\pi e \sigma^2)
\]

- Conditioned on \(Y\), the random variable \(X\) has a Gaussian distribution with mean \(\mathbb{E}[X|Y]\) and variance

\[
\text{Var}(X|Y) = \text{Var}(X) - \frac{\text{Cov}^2(X, Y)}{\text{Var}(Y)} = (1 - \rho^2)\sigma^2
\]

Thus, the conditional entropy is

\[
h(X|Y) = \frac{1}{2} \log(2\pi e \sigma^2 (1 - \rho^2))
\]

- Adding these together yields the joint entropy

\[
h(X, Y) = h(X|Y) + h(Y) = \log \left(2\pi e \sigma^2 \sqrt{1 - \rho^2} \right)
\]

- Taking the difference yields the mutual information

\[
I(X; Y) = h(X) - h(X|Y) = -\frac{1}{2} \log(1 - \rho^2) = \frac{1}{2} \log \left(\frac{1}{1 - \rho^2} \right)
\]

- Note that if \(\rho = \pm 1\) then \(X = Y\) and the mutual information is positive infinity!

- **Example:** (Multivariate Gaussian Distribution) Let \(X^n \sim N(0, K)\) be an \(n\)-dimensional Gaussian vector with mean zero and covariance \(K\). The differential entropy of \(X^n\) is given by

\[
h(X^n) = \frac{n}{2} \log \left(2\pi e |K|^{\frac{1}{n}} \right)
\]

where \(|K|\) denotes the determinant of \(K\). Note that \(|K|^{\frac{1}{n}}\) is the geometric mean of the eigenvalues of \(K\).
7.3 Properties of Differential Entropy

- **Lemma:** Differential entropy satisfies:
  - \( h(X + c) = h(X) \)
  - \( h(aX) = h(X) + \log |a| \) for \( a \neq 0 \).
  - \( h(AX) = h(X) + \log |\text{det}(A)| \) when \( A \) is a square matrix.

- **Proof of scaling property for scalar setting.**
  - The differential entropy of a continuous random variable with density \( f_X(x) \) is
    \[
    h(X) = \mathbb{E}[-\log f_X(X)]
    \]
  - For \( a > 0 \), the cdf of \( Y = aX \) is given by
    \[
    F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[aX \leq y] = F_X(y/a)
    \]
    and thus the density of \( Y \) is
    \[
    f_Y(y) = f_Y'(y) = \frac{d}{dy} F_X(y/a) = \frac{1}{a} f_X(y/a)
    \]
  - As a consequence
    \[
    h(aX) = h(Y)
    = \mathbb{E}[-\log f_Y(Y)]
    = \mathbb{E}[-\log \left( \frac{1}{a} f_X(Y/a) \right)]
    = \mathbb{E}[-\log \left( \frac{1}{a} f_X(Y) \right)]
    = \mathbb{E}[-\log f_X(Y)] + \log a
    = h(X) + \log a
    \]

- **Theorem:** (Gaussian distribution maximizes differential entropy under second moment constraints) The differential entropy of an \( n \)-dimensional vector \( X^n \) with covariance \( K \) is upper bounded by the differential entropy of the multivariate Gaussian distribution with the same covariance,
  \[
  h(X^n) \leq \frac{1}{2} \log((2\pi e)^n |K|)
  \]
  Equality holds if and only if \( X^n \sim N(0, K) \)

- **Proof:**
  - Let \( Y \) be Gaussian with
    \[
    \mathbb{E}[X] = \mathbb{E}[Y], \quad \text{Cov}(Y) = \text{Cov}(X)
    \]
The relative entropy between \( f_X \) and \( f_Y \) obeys

\[
D(f_X \| f_Y) = \mathbb{E} \left[ \log \left( \frac{f_X(X)}{f_Y(X)} \right) \right]
\]

\[
= -h(X) + \mathbb{E} \left[ \log \left( \frac{1}{f_Y(X)} \right) \right]
\]

\[
= -h(X) + \frac{1}{2} \mathbb{E} \left[ (Y - \mathbb{E}[Y])^T \mathbb{C}ov(Y)^{-1} (Y - \mathbb{E}[Y]) \right] + \frac{n}{2} \log(2\pi |K|^{1/n})
\]

\[
= -h(X) + \frac{1}{2} \mathbb{E} \left[ \text{tr}((Y - \mathbb{E}[Y])^T \mathbb{C}ov(Y)^{-1} (Y - \mathbb{E}[Y])) \right] + \frac{n}{2} \log(2\pi |K|^{1/n})
\]

\[
= -h(X) + \frac{1}{2} \text{tr}(\mathbb{C}ov(Y)\mathbb{C}ov(Y)^{-1}) + \frac{n}{2} \log(2\pi |K|^{1/n})
\]

\[
= -h(X) + \frac{n}{2} + \frac{n}{2} \log(2\pi |K|^{1/n})
\]

\[
= -h(X) + h(Y)
\]

Since relative entropy is nonnegative, we conclude that

\[
h(X) \leq h(Y)
\]

- **Theorem:** If \( X \to Y \to \hat{X} \) form a Markov chain, then

\[
\mathbb{E} \left[ (X - \hat{X})^2 \right] \geq \frac{1}{2\pi e} \exp(2h(X|Y))
\]

- **Proof:**
  - Conditioned on the event \( \{Y = y\} \),

\[
\mathbb{E} \left[ (X - \hat{X})^2 \mid Y = y \right] \geq \mathbb{V}ar(X \mid Y = y)
\]

\[
\geq \frac{1}{2\pi e} \exp(2h(X \mid Y = y))
\]

where the second inequality follows from the fact that entropy of \( X \) conditioned on \( Y = y \) is upper bounded by the entropy of Gaussian random variable with the same variance:

\[
h(X|Y = y) \leq \frac{1}{2} \log(2\pi e \mathbb{V}ar(X \mid Y = y))
\]

- Taking expectation of both sides and applying Jensen’s inequality yields the stated result

- **Theorem:** (Entropy Power Inequality) Let \( X \) and \( Y \) be independent \( n \)-dimensional random vectors such that \( h(X) \), \( h(Y) \) and \( h(X + Y) \) exists. Then

\[
e^{\frac{2}{n}h(X + Y)} \geq e^{\frac{2}{n}h(X)} + e^{\frac{2}{n}h(Y)}
\]

Moreover, equality holds if and only if \( X \) and \( Y \) are multivariate Gaussian with proportional covariances.

- There are many different proofs of the entropy power inequality, which are interesting in their own right. The following Lemma is a special case of the EPI that has a simple self-contained proof.
• **Lemma:** Let \( X_1 \) and \( X_2 \) be independent continuous random variables whose distributions are sign invariant (i.e., \( X_i \) and \( -X_i \) have the same distribution). Then,

\[
h\left(\frac{1}{\sqrt{2}} (X_1 + X_2)\right) \geq \frac{1}{2} (h(X_1) + h(X_2))
\]

• **Proof:**

- For any independent random variables \( X_1 \) and \( X_2 \), we have

\[
h(X_1) + h(X_2) = h(X_1, X_2)
\]

\[
= h\left(\frac{1}{\sqrt{2}} (X_1 + X_2), \frac{1}{\sqrt{2}} (X_1 - X_2)\right)
\]

\[
= h\left(\frac{1}{\sqrt{2}} (X_1 + X_2)\right) + h\left(\frac{1}{\sqrt{2}} (X_1 - X_2)\right) - I\left(\frac{1}{\sqrt{2}} (X_1 + X_2); \frac{1}{\sqrt{2}} (X_1 - X_2)\right)
\]

where the second step holds because the linear transformation applied to the vector \((X_1, X_2)\) has determinant one.

- Because of sign invariance, \((X_1 - X_2)\) and \((X_1 + X_2)\) are equal in distribution and thus

\[
h\left(\frac{1}{\sqrt{2}} (X_1 - X_2)\right) = h\left(\frac{1}{\sqrt{2}} (X_1 + X_2)\right).
\]

Combining with the above expression and noting that mutual information is non-negative gives the stated result.

7.4 **Entropic Central Limit Theorem**

- Let \( X_1, X_2, \ldots \) be i.i.d. random variables with mean \( \mu \) and variance \( \sigma^2 \) and let

\[
S_n = \frac{1}{n} \sum_{i=1}^{n} X_i
\]

\[
Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)
\]

denote the average and normalized average of the first \( n \) terms.

- The **Law of Large Numbers** (LLN) states that \( S_n \) converges almost surely to the mean \( \mu \)

- The **Central Limit Theorem** (CLT) states that \( Z_n \) converges in distribution to Gaussian random variable with mean zero and variance \( \sigma^2 \). In other words, for all \( t \in \mathbb{R} \),

\[
P[Z_n \leq t] \to \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)} \, dx
\]

- Now suppose that the random variables \( X_1, X_2, \ldots \) are drawn iid from a continuous distribution with finite differential entropy \( h(X_i) \). The entropic CLT states that the entropy of the normalized sum \( Z_n \) converges to the entropy of the Gaussian distribution with mean zero and variance \( \sigma^2 \), i.e.

\[
h(Z_n) \to \frac{1}{2} \log(2\pi e \sigma^2)
\]

Furthermore, if \( \{X_i\} \) are not Gaussian, then the sequence \( h(Z_n) \), is strictly increasing

\[
h(X_1) = h(Z_1) < h(Z_2) < \cdots < h(Z_n) < \frac{1}{2} \log(2\pi \sigma^2).
\]