ECE 587 / STA 563: Lecture 10 – Review and Applications

Information Theory
Duke University, Fall 2023

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Last Modified: August 24, 2023

Outline of lecture:

10.1 Point-to-Point Communication

The fundamental problem of communication is that of reproducing at one point, either exactly or approximately, a message selected at another point

— Claude Shannon

10.1.1 Recap of Main Theorems

- **Lossless Source Coding:** For a discrete iid source with pmf $p(x)$, the expected length $\mathbb{E}[\ell(X)]$ of the optimal uniquely decodable D-ary source code satisfies

$$\frac{H(X)}{\log D} \le \mathbb{E}[\ell(X)] < \frac{H(X)}{\log D} + 1$$

  By coding over blocks of length n, the expected number of code symbols per source symbol of the optimal uniquely decodable D-ary source code satisfies

$$\frac{H(X)}{\log D} \le \frac{1}{n}\mathbb{E}[\ell(X^n)] < \frac{H(X)}{\log D} + \frac{1}{n}$$

  More generally, for a stationary ergodic source $X_1, X_2, \dots$, the fundamental limit of compression, measured in bits per source symbol, is given by the entropy rate $H(\mathcal{X})$, computed using the base 2 logarithm.

- **Channel Coding:** For a discrete memoryless channel $p(y \mid x)$, there exists a sequence of rate R block-length n coding schemes with error probability tending to zero provided that R is less than the capacity C, which given by

$$C = \max_{p(x)} I(X;Y).$$

  Conversely, if $R > C$ then the probability off error is bounded away from zero.

- **Gaussian Channel:** For an additive white gaussian noise channel with power constraint P and noise variance N, the capacity is given by

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

- **Lossy Source Coding:** For a discrete iid source with pmf $p(x)$ and bounded distortion measure $d(x, \hat{x})$ there exists a sequence of rate R block-length n coding scheme with distortion satisfying

$$\limsup_{n\to\infty} \mathbb{E}\Big[d(X^n, \hat{X}^n)\Big] \le D$$

provided that R is greater than the rate-distortion function $R(D)$, which is given by

$$R(D) = \min_{p(\hat{x}|x)\,:\,\mathbb{E}\left[d(X,\hat{X})\right]\le D} I(X;\hat{X})$$

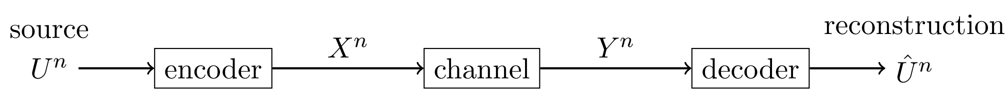
- **Gaussian Source:** For an iid Gaussian source $\mathcal{N}(0, \sigma^2)$ and squared-error distortion $d(x, \hat{x}) = (x - \hat{x})^2$, the rate distortion function is given by

$$R(D) = \begin{cases} \frac{1}{2}\log\left(\frac{\sigma^2}{D}\right), & 0 \le D \le \sigma^2 \\ 0, & D > \sigma^2 \end{cases}$$

10.1.2 Source Channel Separation Theorem

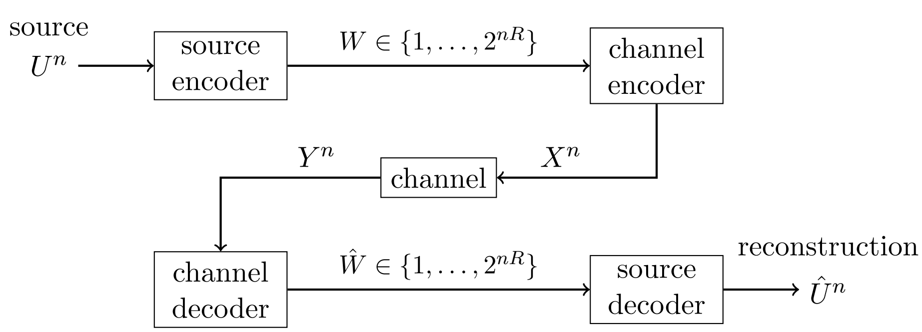
- Suppose we want to communicate an iid source $U_1, U_2, \cdots, U_n$ with n uses of a memoryless channel with capacity C while incurring a distortion no greater than D.

- **Joint source and channel coding:**



  - Encoder: maps source U^n into channel input X^n
  - Decoder: maps channel output Y^n into reconstruction $\hat{U}^n$

- **Separate source and channel coding:**



  - Source encoder: maps source U^n into message $W \in \{1, 2, \cdots, 2^{nR}\}$
  - Channel encoder: maps message W into channel input X^n

- ◦ Channel decoder: maps channel output Y^n into message estimate $\hat{W} \in \{1, 2, \cdots, 2^{nR}\}$
- ◦ Source decoder: maps message estimate into reconstruction $\hat{U}^n$

- **Theorem:** (Source-Channel Separation) Suppose we want to send an iid source with with rate distortion function $R(D)$ across a discrete memoryless channel with capacity C. A distortion D is achievable if and only if

$$C > R(D)$$

Furthermore, there is no loss in using separate source and channel coding.

- **Example with uncoded transmission:** Let $U_1, U_2, \ldots$ be a sequence of iid Gaussian variables with mean zero and variance σ^2. Suppose these values are transmitted over a Gaussian noise channel with noise power N and power constraint P according to the encoding scheme

$$X_i = \frac{\sqrt{P}}{\sigma} U_i$$

such that

$$Y_i = \frac{\sqrt{P}}{\sigma} U_i + Z_i$$

Suppose that the reconstruction of U_i is given by the conditional expectation:

$$\hat{U}_i = \mathbb{E}[U_i \mid Y_i] = \frac{\sqrt{P}\sigma}{P + N} Y_i$$

Then, the squared error distortion of this coding scheme is

$$D = \mathbb{E}\big[(U_i - \mathbb{E}[U_i \mid Y])^2\big] = \frac{\sigma^2}{1 + P/N}$$

Equivalently,

$$\frac{\sigma^2}{D} = 1 + \frac{P}{N} \qquad \Longleftrightarrow \qquad \underbrace{\frac{1}{2}\log\left(\frac{\sigma^2}{D}\right)}_{R(D)} = \underbrace{\frac{1}{2}\log\left(1 + \frac{P}{N}\right)}_{C}$$

Hence, the distortion is precisely the distortion-rate function $D(R)$ of the Gaussian source evaluated at the the capacity of the Gaussian channel.

10.2 Application to Statistical inference

10.2.1 Paramter Estimation

- Suppose that data $X_1, \ldots, X_n$ are drawn i.i.d. from a family of probability measure P_θ indexed by a parameter $\theta \in \Theta$.

- The **minimax risk** associated with a loss function $L(\theta, \hat{\theta})$ is defined by

$$M(\Theta) = \inf_{\delta} \max_{\theta \in \Theta} \mathbb{E}_{X^n \overset{iid}{\sim} P_\theta}[L(\theta, \delta(X^n))]$$

where the infimum is over all estimators $\delta(\cdot)$.

- For any prior distribution π supported on Θ, the minimax risk is bounded from below by the **Bayes risk**, which is defined by

$$B(\pi) = \inf_{\delta} \mathbb{E}_{\theta \sim \pi, X^n \overset{iid}{\sim} P_\theta}[L(\theta, \delta(X^n))]$$

  An estimator the achieves the infimum is called the Bayes rule.

- Suppose that:
  - the "source" defined by π has rate-distortion function $R(D)$ with respect to the loss function $L(\theta, \hat{\theta})$.
  - the mutual information defined by π and $P_\theta^{\otimes n}$ is given by $I(\theta; X^n)$.

  Then, the Bayes risk is bounded from below by the distortion-rate function $D(R)$ evaluated at $I(\theta; X^n)$, i.e.,

$$\text{Bases risk at } \pi \geq D(I(\theta; X^n))$$

  Note that if $I(\theta; X^n)$ can be bounded from above by the capacity of the the "channel" defined by $P_\theta^{\otimes n}$.

10.2.2 Linear Model

- Suppose that data $(X, Y) \in \mathbb{R}^{n \times p} \times \mathbb{R}^n$ depend on an unknown parameter $\beta \in \mathbb{R}^p$ according to the model

$$Y = X\beta + Z, \qquad Z \sim \mathcal{N}(0, \sigma^2 I_n)$$

- Suppose that β is drawn according to a prior distribution π with mean zero and identity covariance.

- The mutual information satisfies

$$\begin{aligned} I(\beta; X, Y) &= I(\beta; Y \mid X) \\ &= h(Y \mid X) - h(Y \mid X, \beta) \\ &= h(Y \mid X) - \frac{n}{2}\log(2\pi e \sigma^2) \end{aligned}$$

  Furthermore, the entropy of $Y \mid X$ is bounded from above by the Gaussian distribution of the same mean and variance, and so

$$h(Y \mid X) \leq \frac{n}{2}\log(2\pi e) + \log\det(\sigma^2 I_n + XX^\top)$$

  Whence,

$$I(\beta; X, Y) \leq \frac{1}{2}\log\det\Big(I_n + \sigma^{-2} XX^\top\Big)$$